Physics 218C: Lecture notes (15-16)

Geometry (cont.) & Zonal flow

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1. Magnetic shear and Quasi-modes

There are two issues in Geometry: shear, which includes velocity shear and magnetic shear, and toroidicity. The study on magnetic shear is relatively older, while people started realizing the importance of velocity shear to magnetic confinement fusion more recently. Toroidicity means ballooning. In lecture 15, ballooning will be briefly introduced in a more physical way than ballooning mode representation.

1.1. Revisiting magnetic shear

In magnetic shears, the safety factor q, which is the pitch of the field lines, varies with radius r. If we expand the gradient operator along the field lines near a resonant surface, we can get

$$\nabla_{\parallel}=\frac{\partial}{\partial z}+\frac{r_s}{Rq(r)}\frac{\partial}{r_s\partial\theta}=\frac{\partial}{\partial_z}+\left(\frac{r_s}{Rq(r_s)}+\frac{x}{L_s}\right)\frac{\partial}{\partial y}, \qquad (1-1)$$

where $1/L_s=-\hat{s}/Rq, dy=r_s d\theta$.

In lecture 14, we constructed shearing coordinates for velocity shear. The point of shearing coordinates is transforming the coordinates into a comoving frame in which shear flow is stationary. By doing this, the total time evolution d/dt goes to a partial time derivative $\partial/\partial t$ with respect to time. We can do the same thing for magnetic shear. Now replacing time t by distance z, and v_y' by $1/L_s$, we obtain

$$\begin{cases} k_x\to k_x'-k_y'z'/L_s \\ k_y\to k_y' \\ k_z\to k_z'-k_y'x'/L_s \end{cases}. \qquad (1-2)$$

N.B. here, z is the distance along the field line. But z is no longer periodic since most of the magnetic surfaces are irrational. Field lines on irrational surfaces extend to infinity, and fill the whole surface. Then, according to the Eikonal equation, we have

$$\frac{dk_x}{dz}=-\frac{k_y}{L_s}, \qquad (1-3)$$

which is the counterpart of Eikonal equation $dk_x/d_t=-k_y v_y'$ in velocity shear. In lecture 14, we can use Eq (1-3) to derive the correlation length along stochastic magnetic field, which satisfies

$$\frac{1}{l_{\parallel c}}\sim\left(k_y^2 D/3L_s^2\right)^{1/3}, \qquad (1-4)$$

where D is something like Rechester and Rosenbluth diffusivity. Eq (1-2) also tells us that $dy'/dz'=x'/L_s$, which means the mode/cell must twist (just like eddy tilting). And twists are different at different radii.

In addition, in terms of velocity shear, if we add a mesoscopically fluctuating shear flow, which refers to something like zonal flow, to a mean shear flow, then we can get a Langevin equation

$$\frac{dk_x}{dt} = -\frac{\partial}{\partial x}\left(k_x\langle v_y\rangle + k_y\tilde{v}_y\right). \qquad (1-5)$$

Eq (1-5) means k_x^2 will diffuse in the k-space and its diffusivity is

$$D_{\boldsymbol{k}} = \sum_q q^2 k_y^2 \left|\tilde{v}_{yq}\right|^2 \tau_{c\boldsymbol{k},q}\,. \qquad (1-6)$$

This is effectively a kind of stochastic shearing effect. Naturally, we can generalize this result to magnetic shear. We add a random component called zonal field to the RHS of Eq (1-3), then get

$$\frac{dk_x}{dz} = -\frac{k_y}{L_s} - \frac{\partial}{\partial x}\left(k_y\frac{\tilde{B}_y}{B_0}\right), \qquad (1-7)$$

which brings us to the same thing as Eq (1-5). Similarly, this will introduce a diffusion of k_x^2 and the diffusivity is

$$D_{\boldsymbol{k}} = \sum_q q^2 k_y^2 \left|\frac{\tilde{B}_{yq}}{B_0}\right|^2 l_{c\boldsymbol{k},q} \qquad (1-8)$$

So how do we get the zonal field? Recall that we get zonal flow by considering the Reynolds stress, in other words, the flux of polarization charge. As a reminder, we have

$$\partial_t\langle\nabla_r^2\phi\rangle + \partial_r\left\langle\tilde{v}_r\nabla_\perp^2\tilde{\phi}\right\rangle = \mu\nabla_r^2\langle\nabla_r^2\phi\rangle,$$

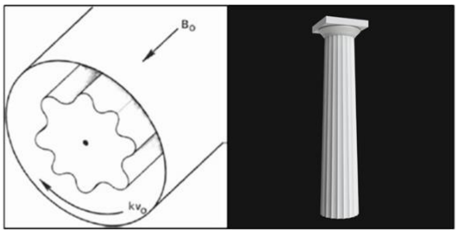
where $\left\langle\tilde{v}_r\nabla_\perp^2\tilde{\phi}\right\rangle = \partial_r\langle\tilde{v}_r\tilde{v}_\theta\rangle$ (azimuthal symmetry is utilized here). So in the same spirit, we can get zonal field by looking at the Ohm's law. More specifically, we have

$$\partial_t\langle A\rangle = -\partial_r\left[\left\langle\tilde{v}_r A_\parallel\right\rangle + \left\langle\tilde{v}_d A_\parallel\right\rangle\right] \qquad (1-9)$$

After relating $A_\parallel$ to $\tilde{v}_r$ through quasi-linear theory and taking the derivative of Eq (1-9), we obtain an equation for the zonal field.

1.2. Resistive interchange mode with magnetic shear

Above discussion brings us to resistive interchange mode. When there is no magnetic shear or resistivity, $k_\parallel = 0$. Interchange mode is also called flute instability because in cylindrical configuration the isopycnic surface looks like the rippling surface of an ancient Greek flute column, as shown in Fig 1.



When there is shear, interchange will be stabilized as $k_\parallel = k_y x/L_s$ near resonance

Fig 1 Flute instability & ancient Greek flute column

surfaces. This comes from the fact that $E_\parallel = 0$. Neutrality condition, Ohm's law of ideal MHD and pressure equation tell us

$$\begin{cases} \nabla_\parallel J_\parallel + \nabla_\perp\cdot\boldsymbol{J}_\perp = \nabla_\parallel J_\parallel - \partial_t\nabla_\perp^2\tilde{\phi} + \kappa\partial_y\tilde{p} \\ E_\parallel = -\partial_t\tilde{A}_\parallel - \nabla_\parallel\tilde{\phi} = 0 \\ J_\parallel = \nabla_\perp^2 A_\parallel \\ \partial_t\tilde{p} + \tilde{v}_r\partial_r\langle p\rangle = 0 \end{cases}. \qquad (1-10)$$

Then by combining these equations, we have

$$\gamma^2\nabla_\perp^2\tilde{\phi} = -k_\parallel\nabla_\perp^2 k_\parallel\tilde{\phi} - \kappa k_y^2\tilde{\phi}\partial_r\langle p\rangle, \qquad (1-11)$$

$$\gamma^2 = \frac{-\int (\nabla_\perp k_\parallel \tilde{\phi})^2 d^3r + \int \kappa k_y^2 \partial_r \langle p \rangle |\tilde{\phi}|^2 d^3r}{\int (\nabla_\perp \tilde{\phi})^2 d^3r}. \quad (1-12)$$

So a non-zero $k_\parallel$ can stabilize the system.

Then, if we introduce resistivity, field and fluid will be decoupled. In electrostatic limit ($\partial_t A_\parallel + \nabla_\parallel \hat{\phi} = \eta \hat{J}$), the vorticity equation now is

$$\nabla_\perp^2 \tilde{\phi} + \frac{v_A^2}{\gamma\eta} \nabla_\parallel^2 \tilde{\phi} + \frac{g}{|L_\rho|\gamma^2} k_y^2 \tilde{\phi} = 0. \quad (1-13)$$

Since $k_\parallel = k_y x / L_s$, Eq (1-13) looks like a quantum harmonic oscillator. The scaling of the growth rate is

$$\gamma \sim O(\eta^{1/3}) \sim O(1/s^{1/3}), \quad (1-14)$$

as $1/s$ is the Lundquist number and it is small.

By the way, the spatial width of resistive interchange w is proportional to $a/s^{1/3}$, and thus is small, too. This is because magnetic shear forces the mode to stick near the resonant surface.

1.3. Roberts and Taylor: twisted slicing modes

Resistive interchange mode is first studied by Furth, Killeen, and Rosenbluth. The G-modes they found are normal modes in which the effect of resistivity is concentrated in a thin region near resonant surfaces where $\boldsymbol{k} \cdot \boldsymbol{B} = 0$. This thin region is similar to the boundary layer in hydrodynamics. But Roberts and Taylor adopted a different perspective, which leads to an alternative class of unstable modes in which the influence of resistivity is not localized. Roberts and Taylor approached the final model, which is a plasma slab of infinite extent in the z direction with finite resistivity and finite shear, through a series of simpler but related problems: 1, interchange of ideal plasma in a sheared magnetic field with resistively coated end plates (localized resistivity → no line-tying effect); 2, resistive interchange in a uniform magnetic field with perfectly conducting end plates; 3, resistive interchange in a sheared magnetic field without line-tying effect.

1.3.1. The gravitational model

The full set of equations of this model is

$$\begin{cases} \rho_0 \dfrac{\partial \boldsymbol{v}}{\partial t} = -\nabla p + \dfrac{1}{4\pi}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B_0} + \dfrac{1}{4\pi}(\nabla \times \boldsymbol{B_0}) \times \boldsymbol{B} + \rho \boldsymbol{g} \\ \dfrac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B_0}) + \dfrac{\eta}{4\pi}\nabla^2 \boldsymbol{B} = (\boldsymbol{B_0} \cdot \nabla)\boldsymbol{v} - (\boldsymbol{v} \cdot \nabla)\boldsymbol{B_0} + \dfrac{\eta}{4\pi}\nabla^2 \boldsymbol{B} \\ \dfrac{\partial \rho}{\partial t} = -(\boldsymbol{v} \cdot \nabla)\rho_0 = -v_x \alpha \rho_0 \\ \nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \cdot \boldsymbol{v} = 0 \end{cases}, \quad (1-15)$$

where $\boldsymbol{B_0} = (0, sxB_0, B_0)$ (as long as there is magnetic shear), $\boldsymbol{g} = -g\hat{\boldsymbol{x}}$. The fluid is contained between perfectly conducting rigid walls at $x = \pm H$, where the boundary conditions for perturbed variables are $v_x = B_x = E_x = E_y = 0$. The shear is weak, i.e., $sH \ll 1$.

1.3.2. Ideal interchange in a sheared magnetic field without line-tying effect

In this section, plasm is assumed to be ideal and confined by two perfectly conducting plane coated with a thin perfectly insulating layer at $z = \pm L$. At these two plates, the boundary conditions are $B_z = v_z = (\nabla \times \boldsymbol{B})_z = 0$.

As a preliminary, consider a system without shear. If we restrict our focus on modes

with $k_z = 0$, then Eq (1-15) show that $\boldsymbol{B} = 0$ and $v_z = 0$.
By taking the curl of the momentum equation, we get

$$\rho_0 \frac{\partial \nabla \times \boldsymbol{v}}{\partial t} = \frac{1}{4\pi}[(\boldsymbol{B_0} \cdot \nabla)\nabla \times \boldsymbol{B} - (\nabla \times \boldsymbol{B_1}) \cdot \nabla \boldsymbol{B_0} - (\nabla \times \boldsymbol{B_0}) \cdot \nabla \boldsymbol{B}] + \nabla \rho \times \boldsymbol{g} \qquad (4-16)$$

The curl of Eq (1-16) gives us

$$-\rho_0 \frac{\partial}{\partial t} \nabla^2 \boldsymbol{v} = -\frac{1}{4\pi}(\boldsymbol{B_0} \cdot \nabla)\nabla^2 \boldsymbol{B} - \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\rho \boldsymbol{g}. \qquad (1-17)$$

Utilizing the continuity equation and Ohm's law, we have

$$\begin{cases} \rho = -\dfrac{v_x \alpha \rho_0}{p} \\ B_x = \dfrac{(\boldsymbol{B_0} \cdot \nabla) v_x}{p}, \end{cases} \qquad (1-18)$$

where we have assumed a time dependence $\exp(pt)$.
Combining Eq (1-18) with Eq (1-17), we get

$$-\rho_0 p \nabla^2 v_x = -\frac{1}{4\pi p}(\boldsymbol{B_0} \cdot \nabla)\nabla^2 (\boldsymbol{B_0} \cdot \nabla) v_x - \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\frac{v_x \alpha \rho_0}{p} g \qquad (1-19)$$

Because $s = 0$, $k_z = 0$, $(\boldsymbol{B_0} \cdot \nabla) v_x = 0$ and $\partial_z^2 v_x = 0$, then we can obtain

$$p = (\alpha g)^{1/2} |k_y| / (k_x^2 + k_y^2). \qquad (1-20)$$

What if the magnetic field is sheared and $k_z \neq 0$? We can rewrite Eq (1-19) as

$$\rho_0 p^2 \nabla^2 v_x = \frac{B_0^2}{4\pi}\left(\frac{\partial}{\partial z} + sx\frac{\partial}{\partial y}\right)\nabla^2\left(\frac{\partial}{\partial z} + sx\frac{\partial}{\partial y}\right) v_x + \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) v_x . \qquad (1-21)$$

Eq (1-21) can be solved in the limit $B_0 \to \infty$ by expanding v_x in powers of $1/B_0^2$. That is we write $v_x = u_0 + u_1 + \cdots$. Then

$$D_0 u_0 = 0, \qquad D_1 u_1 = D_1 u_0, \qquad \ldots \quad , \qquad (1-22)$$

where

$$D_0 = B_0^2 \left(\frac{\partial}{\partial z} + sx\frac{\partial}{\partial y}\right)\nabla^2\left(\frac{\partial}{\partial z} + sx\frac{\partial}{\partial y}\right), \qquad (1-23)$$

and

$$D_1 = \rho_0 \left[p^2 \nabla^2 - \alpha g \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\right], \qquad (1-24)$$

are self-adjoint operators. Then, by transforming the coordinates into twisted coordinate system, i.e., $\xi = x, \chi = y - sxz, \zeta = z$, according to Eq (1-2), we have

$$\left(\frac{\partial}{\partial z} + sx\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial \zeta}, \qquad (1-25)$$

and ξ, χ are constant along field lines. Therefore, any function $f(\xi, \chi)$ which is independent of ζ is then a solution of $D_0 u_0 = 0$ as $(\partial_z + sx\partial_y)h(\xi) = 0$ and $(\partial_z + sx\partial_y)j(\chi) = 0$. However, $D_0 u_1 = D_1 u_0$, can possess a solution u_1 only if $D_1 u_0$ is orthogonal to all solutions of $D_0 u_0 = 0$. Because

$$\int_{-L}^{L} v_0 D_0 u_1 d\zeta = \int_{-L}^{L} v_0 D_1 u_0 d\zeta = \int_{-L}^{L} D_0 v_0 u_1 d\zeta = 0. \qquad (1-26)$$

In Eq (1-26) we utilize the fact that D_0 and D_1 are self-adjoint. Eq (1-26) imposes a further constraint on u_0 which must now satisfy equations

$$D_0 u_0 = 0, \int_L^L D_1 u_0 d\zeta = 0\,. \qquad (1-27)$$

Solutions of Eq (1-27) exist of the form $u_0 = u_0(\xi) \exp ik\chi$ provided

$$\frac{d^2 u_0}{d\xi^2} + (A\xi^2 + B)u_0 = 0, \qquad (1-28)$$

where

$$A = k^2 s^2 (\alpha g/p^2 - 1), B = k^2(\alpha g/p^2 - 1 - 1/3s^2L^2). \qquad (1-29)$$

Because $u_0 = 0$ at $x = \pm H$, $u_0^{-1} d^2 u_0/d\xi^2$ must be negative at some places, which means $s^2(\alpha g/p^2 - 1)\xi^2 + (\alpha g/p^2 - 1 - 1/3s^2L^2) > 0$ at some places. Therefore

$$\alpha g/p^2 - 1 > \frac{1}{3} s^2 L^2/(1 + s^2\xi^2) \geq \frac{1}{3} s^2 L^2/(1 + s^2 H^2). \qquad (1-30)$$

We can conclude that $A > 0$, which allow us to rewrite (1-28) as Weber's equation

$$\frac{d^2 u_0}{dw^2} + \left(\frac{1}{4} w^2 - a\right) u_0 = 0. \qquad (1-31)$$

For any finite value of s, k, H, L, it is possible to find a real positive value of $p^2/\alpha g$ such that the solution of Eq (1-31) satisfies the boundary conditions. In other words, the system is always unstable for $\alpha g > 0$. Therefore, the conclusion is in a system of finite length with perfect conductivity but in which lines of force are not tied at the ends, there are instabilities even when the shear is sufficient to stabilize the corresponding infinitely long system. Magnetic shear simply twisted flux tubes to conform to the field lines. The growth rate is lowered because part of the free energy is converted to rotational kinetic energy, but the stability criterion is unaltered.

1.3.3. Resistive interchange in a uniform magnetic field

In this case, at $z = \pm L$, end plates are perfectly conductive, which means there is line-tying effect. The corresponding boundary condition is $B_z = v_x = v_y = 0$ and v_z is small at $z = \pm L$. Assuming all the perturbed quantities have the form $\exp[pt + i(k_x x + k_y y + k_z z)]$, and setting $k^2 = k_x^2 + k_y^2 + k_z^2$, $\tilde{k}^2 = k_y^2 + k_z^2$, Eq (1-17) and Ohm's law yield

$$p^3 + \frac{\eta k^2 p^2}{4\pi} + \left(\frac{B_0^2 k_z^2}{4\pi\rho_0} - \alpha g \frac{\tilde{k}^2}{k^2}\right) p - \frac{\alpha g \eta \tilde{k}^2}{4\pi} = 0. \qquad (1-32)$$

If $\beta \ll 1$ and $p < c_A k_z$, the approximate dispersion relation is

$$p^2 + \frac{B_0^2 k_z^2}{4\pi\rho_0} p - \frac{\alpha g \eta \tilde{k}^2}{4\pi} = 0. \qquad (1-33)$$

When p is small, one root of Eq (1-33) is

$$p \sim (\alpha g \rho_0 / B_0^2 k_z^2)\eta \tilde{k}^2 \sim (\alpha g \rho_0 L^2/B_0^2)\eta \tilde{k}^2. \qquad (1-34)$$

We can assume $k_x, k_z \ll k_y$, then p is almost independent of k_x and proportional to k_y^2. The physical picture is that alternate thin vertical layers, which are parallel to the magnetic field, are moving up and down, as shown in Fig 2 (vertical layers are marked in green and yellow, green means moving up and yellow means moving down).

The reason why fluids can move across the field is the diffusion of field line as a result of resistivity. This mechanism was first proposed in Furth, Killeen and Rosenbluth. But one may notice that here p is proportional to η instead of $\eta^{1/3}$. This result is inconsistent with the growth rate of G-modes. That's because in this model the length of the mode is restricted by the distance between those two end plates at $z = \pm L$. As will be seen in the next section, in an infinitely extended system, L is proportional to $\eta^{-1/3}$. This eliminates the contradiction.

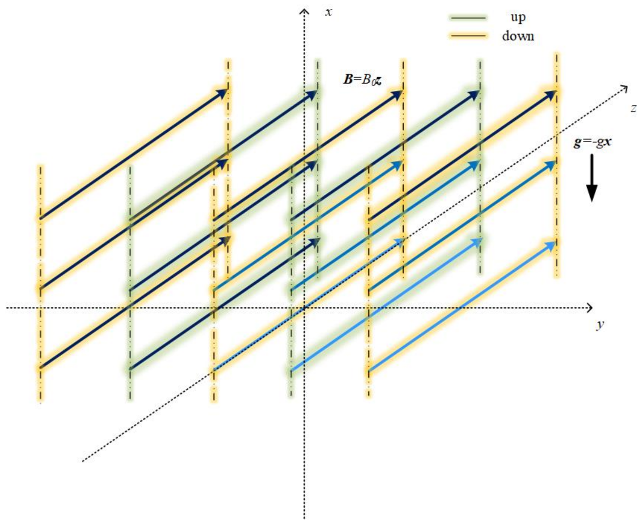


Fig 2 motions of alternate thin vertical layers that are parallel to magnetic field

1.3.4. Twisted slicing modes

Finally, we examine a plasma slab of infinite extent in the z direction, with finite resistivity and finite shear.

Since the presence of shear will not increase the growth rate, $p \ll \eta k^2$, so $\partial \boldsymbol{B}/\partial t$ can be omitted in Ohm's law. In the twisted coordinate, $v_x = v(\zeta)\exp(ik_x\xi + ik_y\chi)$, and Eq (1-21) can be rewritten as

$$\begin{aligned}&(1+\epsilon^2 q)\frac{1}{k_y^2}\frac{\partial^2 v}{\partial \zeta^2} - 2\epsilon^2 q i s\xi\frac{1}{k_u}\frac{\partial v}{\partial \zeta}\\ &-\epsilon^2\left[q(1+s^2\xi^2) + \frac{p^2}{\alpha g}s^2\zeta^2 - \frac{p^2}{\alpha g}\left(\frac{k_x^2}{k_y^2} - 2s\zeta\frac{k_z}{k_y}\right)\right]v = 0\end{aligned}, \qquad (1-35)$$

where $\epsilon^2 = \alpha g\rho_0\eta/pB_0^2$, $q = p^2/\alpha g - 1$. In the limit $1/\epsilon = k_yL \gg 1$, $k_x.k_y \ll 1$, Eq (1-35) reduces to

$$\frac{d^2v}{d\zeta^2} - \frac{p\rho_0\eta}{B_0^2}\left(sk_y\right)^2\zeta^2 v + \frac{p\rho_0\eta k_y^2}{B_0^2}\left(\frac{\alpha g}{p^2} - 1\right)v = 0, \qquad (1-36)$$

which can again be transformed into Weber's equation

$$\left(\frac{d^2}{dw^2} - \frac{1}{4}w^2 + a\right)v = 0, \qquad (1-37)$$

where $\frac{1}{4}w^2 = \frac{\zeta^2}{2\Delta^2}$, $\Delta = \left(\frac{B_0}{sk_y}\right)^{\frac{1}{2}}\left(\frac{1}{p\rho_0\eta}\right)^{\frac{1}{4}}$, $2a = \frac{(p\rho_0\eta)^{\frac{1}{2}}}{sB_0}k_u\left(\frac{\alpha g}{p^2} - 1\right)$. This is like a quantum harmonic oscillator, so its solutions are

$$v = v_n(\zeta) = \exp(-\xi^2/2\Delta^2)H_n\left(\zeta\sqrt{2}/\Delta\right), \qquad (1-38)$$

where H_n are Hermite polynomials. And the growth rate is

$$p_n = \left(\frac{\eta k_\eta^2}{4\pi}\right)^{\frac{1}{3}}(\alpha g)^{\frac{2}{3}}\left(\frac{4\pi\rho_0}{B_0^2 s^2}\right)^{\frac{1}{3}}(2n+1)^{-\frac{2}{3}}, \qquad (1-39)$$

which is the same as that found by Furth, Killeen and Rosenbluth. That's because now the length of the mode is

$$\Delta_n = \left(\frac{4\pi}{\eta k_y^2}\right)^{\frac{1}{3}}\frac{1}{(\alpha g)^{\frac{1}{6}}}\left(\frac{B_n^2}{4\pi\rho_0}\right)^{\frac{1}{3}}\frac{1}{s^{\frac{1}{3}}}(2n+1)^{\frac{2}{3}} \propto \eta^{-\frac{1}{3}}. \qquad (1-40)$$

As we can see, k_x doesn't show up in both Eq (1-39) and Eq (1-40), provided $k_x \ll$

k_y. Hence, we can replace the x dependence of our modes by any arbitrary dependence $g(x)$ so long as $g(x)$ varies slowly. We thus obtain modes of the form

$$v_x(x,y,z) = g(x)v_n(z)\exp[ik_y(y - sxz)]. \qquad (1-41)$$

This mode is called **twisted slicing mode**. Since v_x varies slowly when $y - sxz = const$, at a specific plane $z = const$, the flow pattern is almost constant along the line $y - sxz = const$, which means "convective rolls" get twisted as one moves along z axis but at the same time the flow velocity also decays slowly away in z because of the term $v_n(z)$. See Fig 3, the projection of each convective roll onto the x-y plane is exactly $y = szx$.

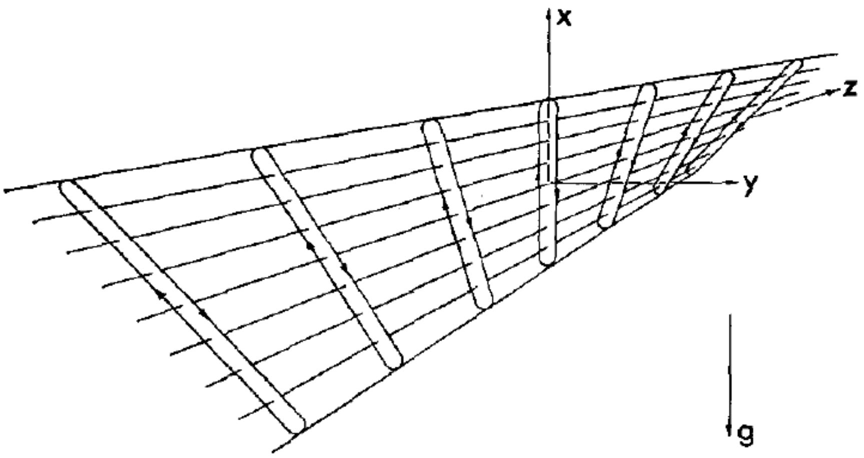


Fig 3 twisted slicing mode

The length of the mode, i.e., Δ, is set by a compromise between (a) rate of release of gravitational potential energy, (b) rate of resistive dissipation, and (c) rate of increase of kinetic energy. In order to reduce (b), flux tubes tend to rotate about vertical axis to align with the local magnetic field as they move up and down. In order to keep this rotational kinetic energy finite, modes must have finite length in z. They achieve this finite length at the expense of some increase in (b). As $\eta \to 0$, $\Delta \to \infty$.

1.3.5. Relation between twisted slicing modes and localized G-modes

As we have fond in Sec 1.3.4., twisted slicing modes have the same growth rate as G-modes. Therefore, there must exist a correlation between these two modes. Actually, twisted slicing modes are just combinations of periodic G-modes at different heights. Because the eigen function of G-modes looks like

$$v_x = \exp(-X^2/2\delta^2)\, H_n(X\sqrt{2}/\delta)\exp(pt + ik_y y + ik_z z), \qquad (1-42)$$

where $sXk_y = sxk_y + k_z$. If we integrate it over k_z (which is equivalent to an integration over x_0), then we get

$$u(x,y,z,t) = \exp(ik_y z)\int f(k_z)dk_z \cdot \exp[ik_z z - (x - x_0)^2/2\delta_0^2]\exp(pt)$$
$$= \delta\sqrt{2}\pi g(x)\exp\left[ik_y(y - sxz) - \left(sk_y z\delta_0\right)^2/2 + p(0)\left(1 + \tfrac{2}{3}(sx)^2\right)t\right], \qquad (1-43)$$

which has the form of the twisted slicing mode.

Of course Eq (1-43) is not an exact normal mode, because it has no precise time dependence. Thus it will disperse after a while. According to Eq (1-43), $\delta p/p = 2/3(sx)^2 {\sim} 10^{-3}$ for a stellarator. Because sx is small, it will behave like a normal mode for a long time. This is why it's called **quasi-mode**. Before the quasi-mode finally disperses, the system can have already entered non-linear regime, which make quasi-mode a good approximation of the normal mode.

For G-modes, $D \sim p/k_x^2 \sim \eta$ while for quasi-modes $D \sim p/k_x^2 \sim \eta^{1/3}$. We can assume the radial correlation length is greatly enhanced, i.e., eddies extend in the x direction. Therefore, quasi-modes can greatly enhance the plasma loss.

2. Toroidicity and ballooning

2.1. A physical approach: Bloch eigenmode equation

In section 1.3., we see that $\phi(z)$ is not eigenmode but quasi-mode, i.e., wave packet. But if we consider poloidal coupling due to toroidicity, wave packets become eigenmodes. That is what we call resistive ballooning mode.

It is worth mentioning that the jargon "ballooning" has double meanings. One meaning corresponds to a physical process driven by curvature. Especially it tends to happen in the "bad curvature" region. Another meaning is more general. People use this word to describe modes/fluctuations where toroidicity matters.

When there is toroidicity, magnetic drifts enter our story. Recall that the Hasegawa-Mima equation is

$$\left(\partial_t - \frac{c}{B_0}\nabla\phi \times \hat{\boldsymbol{z}} \cdot \nabla\right)(\phi - \rho_s^2 \nabla_\perp^2 \phi) + v_* \frac{\partial \phi}{\partial y} = 0. \qquad (2-1)$$

Traditionally, this equation comes from the linearized continuity equation for ions, which is

$$\partial_t \tilde{n} + \tilde{v}_r \partial_r n_0 + n_0 \nabla \cdot \tilde{\boldsymbol{v}} = 0. \qquad (2-2)$$

To couple acoustic wave to drift wave, we need to include $\tilde{v}_\parallel$ into Eq (2-2). Then this leads us to

$$\partial_t \tilde{n} + \tilde{v}_r \partial_r n_0 + n_0 \nabla \cdot \tilde{\boldsymbol{v}} + n_0 \nabla_\parallel \tilde{v}_\parallel = 0. \qquad (2-3)$$

And the parallel velocity of ions satisfies the equation

$$\frac{d}{dt} m_i \tilde{v}_\parallel = -|e| \nabla_\parallel \tilde{\phi}. \qquad (2-4)$$

After linearizing Eq (2-4) and combining it with Eq (2-3), we get the eigenmode equation for drift-acoustic waves, which is

$$-\rho_s^2 \frac{\partial^2}{\partial x^2} \hat{\phi} + k_\theta^2 \rho_s^2 \tilde{\phi} + \left(1 - \frac{\omega^*}{\omega}\right) \hat{\phi} - \frac{k_\theta^2 x^2}{L_s^2 \omega^2} c_s^2 \hat{\phi} = 0 \qquad (2-5)$$

N.B., Eq (2-5) is not a quantum harmonic oscillator, because the factors of $\partial_x^2 \tilde{\phi}$ and $x^2 \tilde{\phi}$ have the same sign. It means instead of a nice mode in a potential well, what we have is a mode in a potential barrier. This mode will drive outgoing waves, which can carry energy. The energy will ultimately be absorbed somewhere. This effect is called magnetic shear damping, which is similar to the damping of oscillating charged particles through radiation. Eq (2-5) gives us the mode structure for drift-acoustic wave, but there will be no instability unless we consider non-adiabatic electrons in sheared system.

Now, instead of the dispersion relation $k_\perp^2 \rho_s^2 + 1 - \omega^*/\omega = 0$ for pure drift wave, the dispersion relation for drift-acoustic wave is

$$k_\perp^2 \rho_s^2 + 1 - \frac{\omega^*}{\omega} - \frac{k_\parallel^2 c_s^2}{\omega^2} = 0. \qquad (2-6)$$

To take into account the effect of toroidicity, i.e., the curvature of magnetic field, besides $\tilde{v}_\parallel$, we need to include the magnetic drift velocity v_{Di} as well. Therefore, Eq (2-3) can be rewritten as

$$\partial_t \tilde{n} + \boldsymbol{v_{Di}} \cdot \nabla \tilde{n} + \tilde{v}_r \partial_r n_0 + n_0 \nabla \cdot \tilde{\boldsymbol{v}} + n_0 \nabla_\parallel \tilde{v}_\parallel = 0 \qquad (2-7)$$

And the expression for drift frequency ω_{Di} is

$$\omega_{Di} = \boldsymbol{v_{Di}} \cdot \nabla = i v_D \big(k_y \cos\theta + k_x \sin\theta\big). \qquad (2-8)$$

Obviously, because the drift frequency depends on poloidal angle θ, it breaks the poloidal symmetry and couples poloidal harmonics. By simply replacing ω by $\omega - \omega_{Di}$ in Eq (2-5), we obtain the eigenmode equation for ballooning mode, which is

$$-\rho_s^2 \frac{\partial^2}{\partial x^2} \hat{\phi}_m + k_\theta^2 \rho_s^2 \hat{\phi}_m + \left(1 - \frac{\omega^*}{\omega}\right) \hat{\phi}_m - \frac{k_\theta^2 x^2}{L_s^2 \omega^2} c_s^2 \hat{\phi}_m$$
$$+\mathrm{T}\left(\hat{\phi}_{m+1} + \hat{\phi}_{m-1}\right) + \mathrm{T}'\left(\hat{\phi}_{m+1} - \hat{\phi}_{m-1}\right) = 0, \qquad (2-9)$$

where $\left(\hat{\phi}_{m+1} + \hat{\phi}_{m-1}\right)$ and $\left(\hat{\phi}_{m+1} - \hat{\phi}_{m-1}\right)$ come from $\cos\theta\, \hat{\phi}_m$ and $\sin\theta\, \hat{\phi}_m$, and linear coupling operators T and T' are of order $\epsilon_{\mathrm{T}} \sim L_n/R$.

Based on Eq (2-9), we can rewrite the eigenmode equation in more compact form as

$$L_{m,n}\hat{\phi}_{m,n} + T_{m+1,n}\hat{\phi}_{m+1,n} + T'_{m-1,n}\hat{\phi}_{m-1,n} = 0. \qquad (2-10)$$

An important fact is that the toroidal mode number n remains as a good quantum number because toroidal symmetry is not broken.

Eq (2-10) is equivalent to a tri-diagonal matrix equation

$$\begin{bmatrix} \cdots & \cdots & 0 & 0 & 0 & 0 \\ T_{m-2} & L_{m-1} & T_m & 0 & 0 & 0 \\ 0 & T_{m-1} & L_m & T_{m+1} & 0 & 0 \\ 0 & 0 & T_m & L_{m+1} & T_{m+2} & 0 \\ 0 & 0 & 0 & T_{m+1} & L_{m+2} & T_{m+3} \\ 0 & 0 & 0 & 0 & \cdots & \cdots \end{bmatrix} \begin{bmatrix} \cdots \\ \phi_{m-1} \\ \phi_m \\ \phi_{m+1} \\ \phi_{m+2} \\ \cdots \end{bmatrix} = 0. \qquad (2-11)$$

Evidently, this matrix reminds us of the one-dimensional linear harmonic oscillator chain (as shown in Fig 4). So the problem reduces to Bloch eigenmode problem and the Bloch eigenfunction is a superposition of linearly coupled poloidal harmonics.

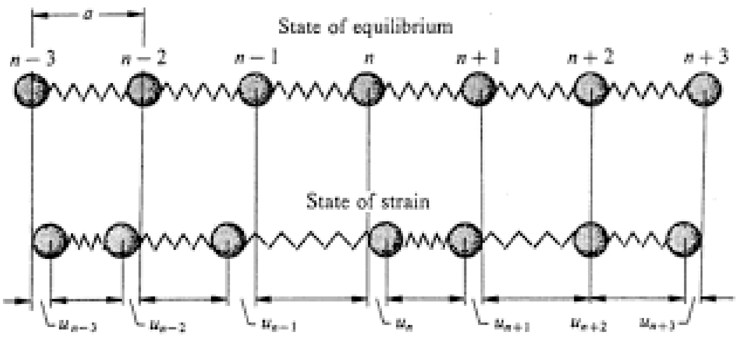


Fig 4 chain of one-dimensional harmonic oscillators

One of the important things in the game of ballooning is that a new scale, the spacing between two adjacent poloidal harmonics. It is easy to get this scale. Assume $q(r_0) = m/n$, and $q(r_0 + \Delta) = (m+1)/n$, after a simple calculation, we have

$$\Delta = \frac{1}{nq'(r_0)} = \frac{1}{k_\theta \hat{s}}. \qquad (2-12)$$

In order for Δr to be much smaller than some macroscopic scales such as L_n and L_p, we always require toroidal mode number n to be large. If this is the case, we may assume there is a translational invariance for two near neighbors, i.e., two adjacent poloidal harmonics look the same. Mathematically, this approximate translational invariance gives us

$$\begin{cases} \hat{\phi}_{m+1} = \hat{\phi}_m(x + \Delta) \\ \hat{\phi}_{m-1} = \hat{\phi}_m(x - \Delta) \end{cases}. \qquad (2-13)$$

Substituting this relation into Eq (1-24), we obtain

$$L_{m,n}\hat{\phi}_{m,n}(x) + T_{m+1,n}\hat{\phi}_{m,n}(x + \Delta) + T'_{m-1,n}\hat{\phi}_{m,n}(x - \Delta) = 0. \qquad (2-14)$$

Now the tridiagonal matrix equation is converted to a differential difference equation.

By taking the Fourier transform in x of Eq (2-14), we can get rid of this spacing shift, which leaves us with a phase factor instead. N.B., the Fourier transform variable of x is η, the distance along field line. That's because $k_{\parallel}$ is linearly related to x. So the Fourier transform of x is indeed equivalent to the Fourier transform of $k_{\parallel}$. By solving Eq (2-14) in η-space, we can determine how mode varies along the field line, which is analogous to quaismode. Basically the mode will be big in the bad curvature region and small in the good curvature region. Like Bloch envelope, we can further determine the radial envelope from envelope boundary condition.

If Δ is much smaller than $\left(\hat{\phi}'/\hat{\phi}\right)^{-1}$, the characteristic length scale of $\hat{\phi}$, then we can expand in Δ. This case is named as "strong ballooning" regime. However, if this condition is not true, we can no longer expand in Δ. We must solve the equation by using Fourier transform. This regime is called "weak ballooning", in which the mode extends in the ballooning coordinate η.

2.2. A mathematical approach: Ballooning Mode Representation

In any axisymmetric toroidal system, the magnetic field can be expressed as

$$\boldsymbol{B} = \nabla\psi \times \nabla\zeta + I(\psi)\nabla\zeta, \qquad (2-15)$$

where $\psi = const$ defines a toroidal magnetic surface and ζ is toroidal angle. So we can construct an orthogonal coordinate system (ψ, ζ, χ) where χ is a poloidal, angle-like coordinate. In toroidal axisymmetric plasmas, some of the most persistent instabilities are those which have short wavelength perpendicular to the magnetic field but long wavelength to it, because bending magnetic field costs too much energy. Since the toroidal symmetry is still kept when we have toroidicity, the appropriate Eikonal form for such oscillations is

$$\phi(\psi, \chi, \zeta) = F(\psi, \chi)\exp\left[in\left(\zeta - \int^{\chi} \nu d\chi'\right)\right], \qquad (2-16)$$

where $\nu = d\zeta/d\chi_0, q = (2\pi)^{-1}\oint \nu d\chi'$ and $n \gg 1$.

According to the definition of ν, if we move along the field line, $d\zeta = \nu d\chi$, the change of phase factor in Eq (2-16) is 0. The small $k_{\parallel}$ are embodied in the slowly varying function $F(\psi, \chi)$. On the contrast, if we move across the field line, since $n \gg 1$, the phase varies quickly.

However, when there is magnetic shear, this expression is contradictory to the requirement of periodicity in the poloidal angle χ for all values of ψ, without abandoning the hypothesis that $F(\psi, \chi)$ varies slowly. Here I can give a simple proof of this statement. If we assume $\phi(\chi = 0) = \phi(\chi = 2\pi)$, then we must have

$$F(\psi, 0) = F(\psi, 2\pi)\exp[-i2\pi nq(\psi)]. \qquad (2-17)$$

We have already assumed that F varies slowly with respect to ψ while $\exp[-i2\pi nq(\psi)]$ is a quickly varying function of χ as $n \gg 1$. Therefore, Eq (2-17) cannot hold unless we assume F also varies quickly, which contradicts the whole concept of an eikonal representation. To reconcile periodicity and shear, Connor, Hastie, and Taylor constructed the following representation in 1979.

If $\hat{\phi}(\eta, x)$ is a solution of a two-dimensional eigenvalue problem

$$\mathcal{L}(\eta, x)\hat{\phi}(\eta, x) = \lambda\hat{\phi}(\eta, x), \qquad (2-18)$$

then its transformation, $\varphi(\theta, x)$, which is defined as

$$\varphi(\theta, x) = \sum_m e^{-im\theta}\int_{-\infty}^{\infty} e^{im\eta}\hat{\phi}(\eta, x)d\eta, \qquad (2-19)$$

is also a solution to the same differential equation with the same eigenvalue, i.e.,

$$\mathcal{L}(\theta, x)\varphi(\theta, x) = \lambda\varphi(\theta, x). \qquad (2-20)$$

N. B., here $\varphi(\theta,x)$ and $\mathcal{L}$ are periodic in θ, $0 \leq \theta \leq 2\pi$ while $\hat{\phi}$ is not periodic and is defined in the infinite domain $-\infty < \eta < \infty$.

Eq (2-19) is called **Ballooning Mode Transformation**. This transformation contains three successive steps:

- First, since $\varphi(\theta,x)$ is periodic in θ, we can get its Fourier expansion, which is
$$a_m = \frac{1}{2\pi}\oint e^{im\theta}\varphi(\theta)d\theta\,. \qquad (2-21)$$
- Second, we can extend the domain of definition of a_m from integer number to real number. For any well-behaved function $\varphi(\theta)$, the generalization of a_m is
$$a(s) = \frac{1}{\pi}\sum_m a_m \frac{\sin[(m-s)\pi]}{(m-s)}. \qquad (2-22)$$
- Finally, by taking the inverse Fourier transformation of $a(s)$, we can obtain $\hat{\phi}(\eta,x)$.

We can prove that if $\mathcal{L}$ is a differential operator with periodic coefficients, then $\phi(\theta)$ will be a periodic solution of $\mathcal{L}(\partial/\partial\theta)\varphi(\theta) = \lambda\varphi(\theta)$ provided $\hat{\phi}$ is a solution of $\mathcal{L}(\partial/\partial\eta)\hat{\phi}(\eta) = \lambda\hat{\phi}(\eta)$ in the infinite domain $-\infty < \eta < \infty$. Substituting Eq (2-19) into Eq (2-20), we get

$$\begin{aligned}(\mathcal{L}(\partial/\partial\theta) - \lambda)\varphi(\theta) &= (\mathcal{L}(\partial/\partial\theta) - \lambda)\textstyle\sum_m e^{-im\theta}\int_{-\infty}^{\infty} e^{im\eta}\hat{\phi}(\eta)d\eta \\ &= \textstyle\sum_m(-im-\lambda)e^{-im\theta}\int_{-\infty}^{\infty} e^{im\eta}\hat{\phi}(\eta,x)d\eta \qquad (2-23)\\ &= \textstyle\sum_m e^{-im\theta}\int_{-\infty}^{\infty}(-im-\lambda)e^{im\eta}\hat{\phi}(\eta)d\eta \\ &= \textstyle\sum_m e^{-im\theta}\int_{-\infty}^{\infty} e^{im\eta}(L(\partial/\partial\eta)-\lambda)\hat{\phi}(\eta)d\eta = 0.\end{aligned}$$

The last step is achieved through partial integration. Clearly, when $L(\partial/\partial\eta)\hat{\phi}(\eta) = \lambda\hat{\phi}(\eta)$, $\mathcal{L}(\partial/\partial\theta)\varphi(\theta) = \lambda\varphi(\theta)$. Then, we can solve this differential equation by using the eikonal form of $\hat{\phi}(\eta,x)$ without considering the requirement of periodicity. Once we get $\hat{\phi}(\eta,x)$, we can immediately get $\varphi(\theta,x)$ through Ballooning Mode Transformation.

Here comes a question, how do we interpret the physical meaning of the coordinate η? η is extended poloidal coordinate (field line position), more specifically, the distance along the field line. Recalling Eq (2-16), because $\hat{\phi}$ is no longer periodic, after ignoring the ignorable coordinate ζ, the expression for $\hat{\phi}$ is

$$\hat{\phi}(\eta,x) = F(x,\eta)\exp\left[-in\int^{\chi}\nu d\eta'\right]. \qquad (2-24)$$

Plugging this expression into Eq (2-19), we obtain

$$\varphi(\theta,x) = \sum_m e^{-im\theta}\int_{-\infty}^{\infty}F(x,\eta)\exp\left[im\eta - in\int^{\eta}\nu d\eta'\right]d\eta\,. \qquad (2-25)$$

As is mentioned above, $\nu = d\zeta/d\eta \approx q$ (here we exclude spherical tokamak), so we can rewrite Eq (2-24) as

$$\varphi(\theta,x) = \sum_m e^{-im\theta}\int_{-\infty}^{\infty}F(x,\eta)\exp[i(m-nq)\eta]\,d\eta\,. \qquad (2-26)$$

Because $(m-nq) = k_{\parallel}$, η is reciprocal to $k_{\parallel}$, and thus the physical meaning of η is the distance along field line.

We can look at the structure of the solution $\varphi(\theta,x)$ in another way. By assuming appropriate convergence properties, we can change the sequence of integral and summation in Eq (2-19), and get

$$\varphi(\theta,x)=\int_{-\infty}^{\infty}\sum_m \exp[-im(\theta-\eta)]\hat{\phi}(\eta,x)d\eta \qquad (2-27)$$

Since $\sum_m \exp\big(-im(\theta-\eta)\big)=\sum_N \delta(\theta-\eta-2\pi N)$, we can rewrite Eq (2-24) as

$$\varphi(\theta,x)=\sum_N \hat{\phi}(\theta-2\pi N,x)\,. \qquad (2-28)$$

As the structure of $\hat{\phi}$ is given by Eq (2-24), we can see φ is indeed an infinite sum of "quasi-modes" as mentioned in Sec. 1.3.

2.3. Effect of toroidicity on drift wave

Recalling equation (2-15), the solution is an outgoing wave, and the wavelength gradually decreases (as shown in Fig 5). At a point, the wave can resonate with ions ($\omega=k_\parallel v_{thi}$) and then be dissipated. This is similar to the radiative damping of oscillating charged particles.

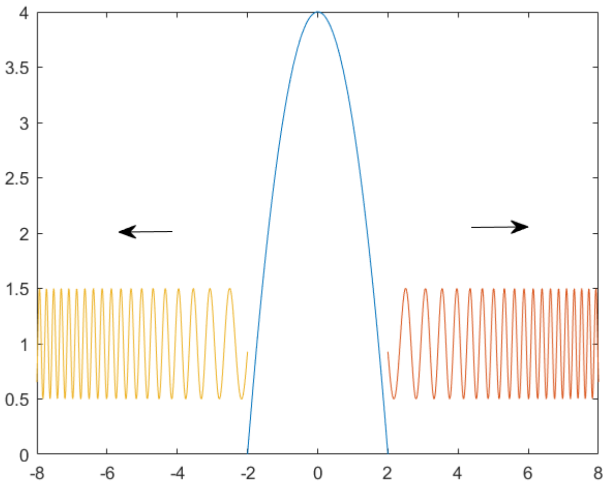


Fig 5 outgoing wave

Then, after including toroidicity and taking the Fourier transform, we get the following eigenmode equation

$$\left[\frac{d^2}{d\eta^2}+\eta_s^2\Omega^2 Q(\Omega,\eta)\right]\hat{\phi}(\eta)=0, \qquad (2-29)$$

where $Q(\Omega,\eta)=b_\theta^2(1+\hat{s}^2\eta^2)+1-1/\Omega+(2\epsilon_n/\Omega)(\cos\eta+\hat{s}\eta\sin\eta)$, $\Omega=\omega_e^*$, $b_\theta=k_\theta^2\rho_s^2$, $\epsilon_n=r_n/R$, and $\eta_s=qb_\theta^{1/2}/\epsilon_n$.

There is no surprise that we still have a potential barrier η^2 in Eq (2-29) because the

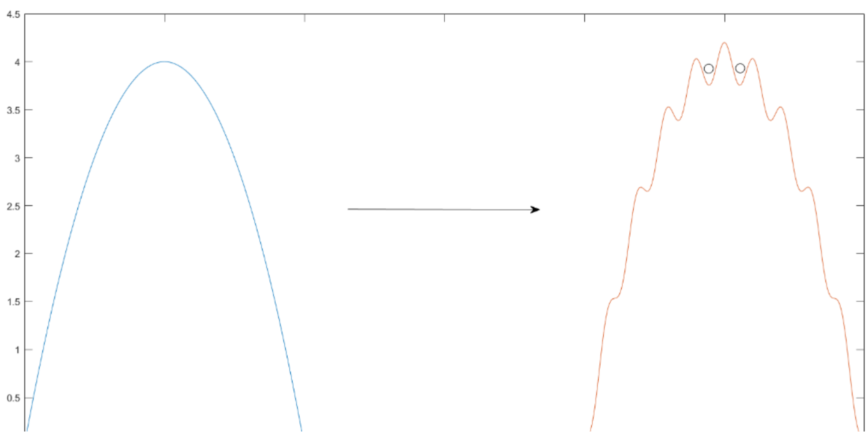


Fig 6 modulation of toroidicity on potential structure

Fourier transform of a Gaussian is a Gaussian. But toroidal-coupling effects introduce modulations on the anti-well potential structures (as shown in Fig 6).
As we can see in Fig 6, the ripples of the potential structure allow some wave energy to be trapped, then we can get standing eigenmode, which are toroidicity induced mode. N. B., this mode is not ballooning mode because they are not driven by curvature. They just indicate us that curvature can allow us to get around radiative damping. In other words, the geometry can actually interfere with the process of coupling to Landau damping (Chen, Liu, and C. Z. Cheng, The Physics of Fluids 23.11 (1980): 2242-2249.). By the way, we can get nearly the same eigenmode equation as Eq (2-29) by using ballooning mode representation (e.g., R. J. Hastie, K. W. Hesketh, and J. B. Taylor, Nucl. Fusion, 19, 1223 (1979)). So in principle, ballooning mode representation can give us the same result as Bloch eigenmode equation.
In these days, flat q regime is one of the important topics in MFE. By controlling the current profile, we can make the shear very weak. This experiment has been done on many famous machines, including JET, TFTR, DIII-D, EAST and KSTAR. As $\hat{s} \rightarrow 0$, we will have fewer resonant surfaces and ballooning structure can fall apart and collapse. Therefore both curvature drive and curvature effects are weakened, leading to confinement improvement.

3. Saturation Mechanisms

Because we are interested in the implications of the relation of the mechanism to the scaling and what determines fluxes $\langle \tilde{v}_r \tilde{n} \rangle$ and intensity $\langle \tilde{v}_r^2 \rangle$, we care about saturation mechanisms.
In reality, tokamak is a strongly driven system with a source and a sink. We can divide a tokamak into three regions: heat deposition region, confinement region and boundary, as shown in Fig 7. There are two names of the game in confinement region. One of the names is what the gradient in the confinement region is, and another name is how high the edge temperature is, which is a key point of H-mode. There should be no surprise, because gradient and edge temperature are just the slope and the intercept of the profile. This configuration is similar to a fixed flux convection problem, i.e., we can control the heating power and let plasma tells us what the temperature is.

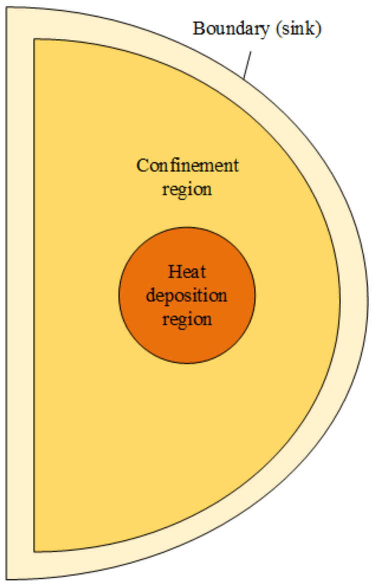


Fig 7 Tokamak: a driven system

Sometimes people quote mixing length theory (MLT) as a saturation. But this is totally wrong. Mixing length limit is the limit at which the non-linear terms become roughly comparable to the linear terms, i.e.,

$$\underline{\tilde{\boldsymbol{v}} \cdot \nabla \tilde{n}} \sim -\underline{\tilde{v}_r \frac{\partial \langle n \rangle}{\partial r}} \qquad (3-1)$$

So what MLT really tells us is we are entering the non-linear regime.
By figuring out where the energy goes, we can coarsely classify different saturation mechanisms into three groups.

3.1. Classification of saturation mechanisms

1) Eliminating free energy or turning off coupling to free energy
Just like the engine will stop when the petrol is used up, instability will stop growing when free energy is used up. One typical example is the formation of plateau in one-

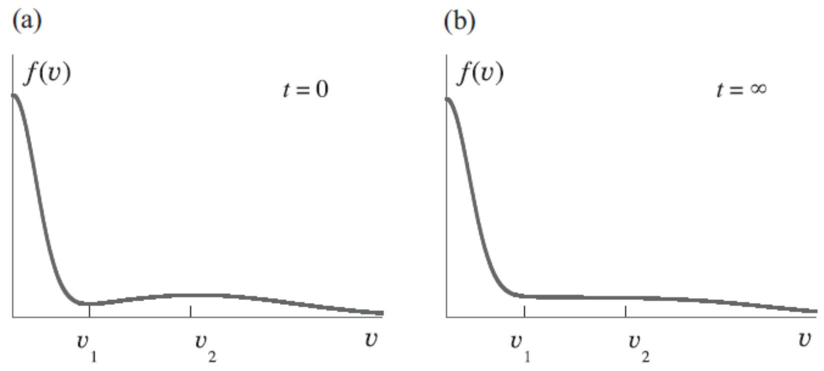


Fig 8 the formation of plateau in distribution function

dimensional quasi-linear theory, i.e., the bump-on-tail problem. Considering the wave-particle interaction, if there is a bump on the tail of particles' distribution function, then the wave can grow by gaining energy from particles. However, as particles gradually lose energy, the bump will be flattened, which means the wave can no longer extract free energy from particles (see Fig 8). Another example is shearing. Shears, such as $\boldsymbol{E} \times \boldsymbol{B}$ shear, can reduce the efficiency of extracting free energy from the system. In energetic particle business, nonlinear frequency shifts may have similar effects. All these examples have the same property that free energy is removed.
2) Coupling to dissipation
One typical example of this kind of saturation is 3D turbulence. If we put energy at large scales by stirring the fluid, then the energy will cascade down to very small scales and finally be dissipated into heat. In this case, the mechanical energy of the fluid is converted to the thermal energy of the fluid. To be specific, assume ϵ, the rate of energy transfer between different scales, is proportional to $v^2(l)/\tau \sim v^3(l)/l$, and dissipation rate at the micro scale is proportional to $\nu v^2(l_d)/l_d^2$, then to balance the energy, we must have

$$\begin{aligned} v(l) &\sim \epsilon^{1/3} l^{1/3} \\ l_d &\sim \frac{\nu}{\epsilon^{1/4}} \end{aligned} \quad . \qquad (3-2)$$

Here the physics is coupling to damping by nonlinear transfer. This is the prototype of all the saturations by non-linear transfer, including mode-mode coupling, nonlinear wave-particle interaction, Compton Scattering, etc. They share the same idea, but they have different kinds of damping.
By the way, we need to realize that weakly damped degree of freedom is most effective at absorbing energy. That's because heavily damped modes are difficult to excite and

couple to. For example, in K41, scales at which $k \gtrsim k_d$ are sinks, not scales at which $k \gg k_d$.

What about inverse cascade? In two-dimensional cases, we have a tendency of the energy to go the other way. In other words, we have some excitations at small scales (at which $k\rho \lesssim 1$), and ultimately, these excitations are locally transferred to large scales. But what is the fate of the energy? For inverse cascade, we need to consider scale independent damping because things like Dk^2 and νk^2 become small at large scales. One potential candidate is drag. In the story of zonal flow, there is some friction of the flow over trapped particles, resulting in some damping like $\mu \sim \epsilon \nu_{i,i}$. When the damping is weak, energy will accumulate at large scales, and then give us a shear flow. This large-scale shear flow will back react on the inverse cascade. Above discussion is limited in k-space. But energy can also be lost at the boundary. Both turbulence spreading and avalanching can redistribute energy in space and often lose it at the boundary. Therefore, in addition to k-space coupling, spatial coupling matters in this type of saturation mechanisms, too.

3) Coupling to "Harmless" degree of freedom

This case specifically refers to zonal mode. To be clear, "damped" and "harmless" are not mutually exclusive. Damped means perturbation decays, and "harmless" means the perturbation is converted to scales which don't degrade confinement. In MFT, a "harmless" mode is a mode with $m, n \equiv 0$, which is precisely zonal mode. In zonal mode, $\tilde{v}_r$ and $\tilde{B}_r$ are 0 so there is no transport. Of course the dissipation of harmless degrees of freedom ultimately disposes of energy. By dumping energy into modes with the proper symmetry, the system can saturate itself.

So in my opinion, system can be saturated by two main mechanisms. First is that the instability can no longer gain free energy, no matter that's because free energy is used up or we turn off the coupling of the instability to free energy. Second can be summarized as transferring the energy of the perturbation to other scales or places. Zonal mode is a more special mechanism because it owns special symmetry so that it is harmless.

3.2. Zonal modes

3.2.1. Basic properties of zonal modes

Zonal modes are modes of thermodynamic variables with poloidal and toroidal symmetry, including potential $\phi(r)$, density $n(r)$, temperature $T(r)$, etc. These thermodynamic variables are important in different cases, as shown in Table 1.

Potential		It is important everywhere, as it produces an $\boldsymbol{E} \times \boldsymbol{B}$ flow
Corrugation	Zonal density	CTEM (BTW, it appears in Hasegawa-Wakatani, but it's weaker)
	Electron temperature	CTEM
	Ion temperature	ITG

table 1 Zonal modes

By generating zonal perturbations, the structure of the distribution function is changed. The reason why zonal modes are important is that zonal mode coupling exploits all three saturation channels introduced in Sec 3.1. First, zonal flow can couple to harmless D-O-F because its poloidal symmetry. Second, zonal modes are weakly damped, so they are coupling to dissipation. Third, zonal modes can remove the coupling to free energy. Specifically, zonal corrugation can add modulations to profile. As shown in Fig 7, if we impose wiggle on the density profile, we can have a staircase. At flat spots of the staircase, perturbations cannot grow because of the lack of free energy. At steep spots, shear flow is spatially co-located with this steepening, so nothing is going to grow there either.

In addition to the above reason, zonal modes have other interesting features. Zonal modes are modes of minimal inertia, which means they are easier to excite. Remember that for zonal modes, we have $\partial_t(k_\perp\rho_s^2)\phi = Drive$ while for drift waves it is $\partial_t(1+k_\perp^2\rho_s^2)\phi = Drive$. For the same drive, zonal modes will have larger ϕ because of smaller inertia. And zonal modes are modes of minimal transport, so they don't degrade confinement. Last but not least, zonal modes are modes of minimal dissipation, which make them easily excited. Therefore, zonal modes are natural repositories for free energy released by micro-instabilities.

Zonal flow can be generated by inhomogeneous PV mixing (i.e., potential vorticity flux) and one direction of symmetry. Recall in lecture 7, the equation for zonal flow is

$$\partial_t\langle\nabla_r^2\phi\rangle + \partial_r\langle\tilde{v}_r\nabla_\perp^2\tilde{\phi}\rangle = \mu\nabla_r^2\langle\nabla_r^2\phi\rangle. \qquad (3-3)$$

Since we have symmetry in y direction, we have

$$\langle\tilde{v}_r\nabla_\perp^2\tilde{\phi}\rangle = \langle\partial_x(\partial_y\tilde{\phi}\partial_x\tilde{\phi})\rangle - \underbrace{\langle\partial_{y,x}\tilde{\phi}\partial_x\tilde{\phi}\rangle}_{odd\ in\ k_y,=0} = \partial_x\langle\partial_y\tilde{\phi}\partial_x\tilde{\phi}\rangle = \partial_r\langle\tilde{v}_r\tilde{v}_\theta\rangle. \qquad (3-4)$$

The Reynolds force in Eq (3-4) can drive zonal flow.

For zonal flow, $k_\theta = 0$. But why is zonal flow more interesting than other low-k_θ modes? Of course low-k_θ modes can provide strong shear, but they can also trigger radial transport, too. Therefore, they involve a trade-off between shearing and enhanced transport while zonal flow is absolutely harmless. What's more, their inertia is larger than zonal flow's and thus harder to excite.

It is very useful to trace the energy flow when things get involved with zonal flow. As shown in Fig 8, free energy is initially stored in electron distribution function, i.e., $1/L_n$ and $1/L_p$. These can make instabilities and thus make waves. The beats of waves produce stresses and flows. N.B., waves are on scales of k while stresses and flows

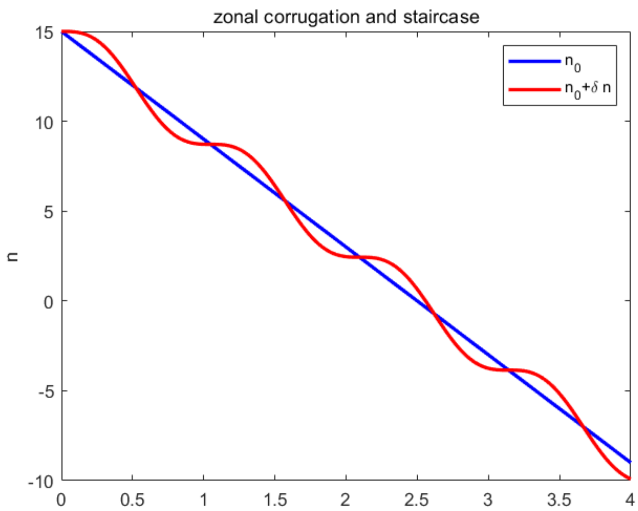


Fig 10 formation of staircase in density profile

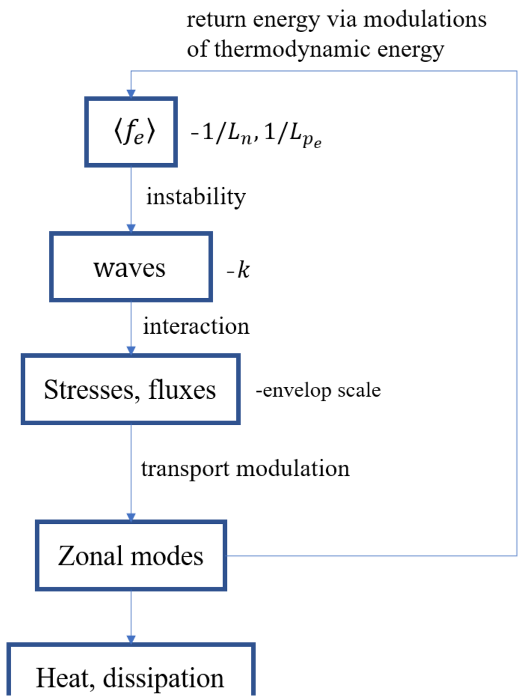


Fig 9 energy flow in EDW or CTEM

are on the envelop scale. Then by the process of transport or modulation, we can obtain zonal modes. Some of zonal modes' energy goes to heat dissipation while the energy sitting in the zonal flow goes back to the distribution function via modulations of thermodynamic quantities.

3.2.2. Modulational instability: a feedback loop

The energy density for electron drift wave or collisionless trapped electron mode is

$$\epsilon = \int d^3x\rho c_s^2\left[\left(\frac{\tilde{n}}{n}\right)^2 + \rho_s^2\left(\nabla_\perp\frac{|e|\hat{\phi}}{T}\right)^2\right]. \qquad (3-5)$$

The evolution of ϵ is

$$\partial_t\epsilon = -\int d^3x[\langle\tilde{v}_r\tilde{n}\rangle\partial_r\langle\delta n\rangle] - \int d^3x\langle v_E\rangle'\langle\tilde{v}_{rE}\tilde{v}_{yE}\rangle - \int d^3x\langle\tilde{\phi}\partial_t h\rangle + \cdots. \quad (3-6)$$

The first term on the right-hand side is the coupling of waves to zonal density, and the second term is Reynolds power, corresponding to the coupling to zonal flow. The third term on the RHS is the drive from non-adiabatic electron distribution.

But how do we calculate the Reynolds power? Since $\tilde{v}_{rE}$ and $\tilde{v}_{yE}$ are just $\boldsymbol{E}\times\boldsymbol{B}$ drift velocities, the relation between $\langle\tilde{v}_{rE}\tilde{v}_{yE}\rangle$ and $\langle k_r k_\theta\rangle$ is

$$\langle\tilde{v}_{rE}\tilde{v}_{yE}\rangle = \left\langle\frac{c^2}{B_0^2}\tilde{E}_\theta\tilde{E}_r\right\rangle = -\frac{c^2}{B_0^2}\sum_{\boldsymbol{k}}\langle k_r k_\theta\rangle|\tilde{\phi}_{\boldsymbol{k}}|^2 \qquad (3-6)$$

To make the correlation $\langle k_r k_\theta\rangle$ non-trivial, we need a radially propagating wave, because $k_r \neq 0$. Here, the velocity shear tends to align k_r, k_θ. Recall that in shearing coordinate, we have

$$\frac{d\boldsymbol{k}}{dt} = -\frac{\partial}{\partial\boldsymbol{r}}\left(\underbrace{\omega}_{=0} + \boldsymbol{k}\cdot\boldsymbol{v}\right), \qquad (3-7)$$

which further gives us

$$\frac{dk_r}{dt} = k_r^{(0)} - k_\theta\langle v_E\rangle' t. \qquad (3-8)$$

With $t \lesssim \tau_c$,

$$\langle k_r k_\theta\rangle = \langle k_r^0 k_\theta\rangle - k_\theta^2\langle v_E\rangle'\tau_c. \qquad (3-9)$$

The physics behind Eq (3-8) is eddy tilting (see Fig 11). Here comes a feedback loop: if we induce a little bit of shear that tilts the eddies, we can obtain a non-trivial $\langle k_r k_\theta\rangle$ according to Eq (3-8). Then zonal flow will be driven by the resulting Reynolds force and tilt eddies further. This is in essence what modulational instability means.

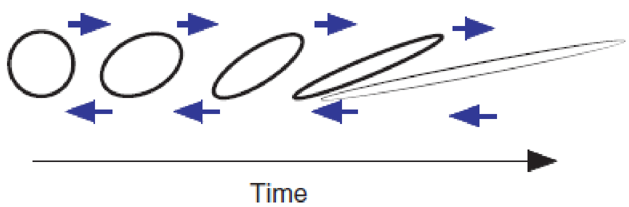


Fig 11 shearing of eddies

3.2.3. Wave kinetics: a systematic approach

We have already got the physical picture of the interaction between waves and zonal flow in Sec 3.2.2. In this section, we want to calculate in detail the response of the wave population to shear under the framework of wave kinetics.

We start this business by exploiting adiabatic invariance. Adiabatic invariants are approximately conserved quantity due to time scale separation. In this problem, the frequency of drift wave ω_{DW} is much higher than the frequency of zonal modes Ω_{ZF}, which is nearly equal to 0. Then the action density, which is defined as

$$N = \epsilon/\omega \tag{3-10}$$

where ϵ is the energy density and ω is the frequency. If ϵ undergoes a slow modulation due to the slow growth of zonal flow, then the action density should be conserved.

Here the action density N is a distribution function of wave vector, position, and time, i.e., $N = N(\boldsymbol{k}, \boldsymbol{x}, t)$, where $\boldsymbol{x}$ and $\boldsymbol{k}$ are Hamiltonian variables. Wave vector $\boldsymbol{k}$ specifies the direction or momentum of the wave, and the position $\boldsymbol{x}$ describes the wave packet position. The physical meaning of N is the population of waves. Remember in quantum mechanics, we have $E = (n + 1/2)\hbar\omega$. Here the action density N is exactly the same thing as the quantum number n. Therefore, there is an analogy between waves and particles. Just as particles satisfy Vlasov equation, waves satisfy

$$\frac{dN}{dt} = \frac{\partial}{\partial t}N + \boldsymbol{v}_{gr} \cdot \nabla N + \frac{d\boldsymbol{k}}{dt} \cdot \nabla_{\boldsymbol{k}} N = 0. \tag{3-11}$$

Just as the characteristic equations of Vlasov equation, the characteristic equations of wave kinetic equation are

$$\begin{aligned} \frac{d\boldsymbol{x}}{dt} &= \boldsymbol{v}_{gr} + \boldsymbol{v} \\ \frac{d\boldsymbol{k}}{dt} &= -\frac{\partial}{\partial \boldsymbol{x}}(\omega + \boldsymbol{k} \cdot \boldsymbol{v}) \end{aligned}. \tag{3-12}$$

Combining equation (3-11) and (3-12), we have

$$\frac{\partial N}{\partial t} + (\boldsymbol{v}_{gr} + \boldsymbol{v}) \cdot \nabla N - \frac{\partial}{\partial \boldsymbol{x}}(\omega + \boldsymbol{k} \cdot \boldsymbol{v}) \cdot \frac{\partial}{\partial \boldsymbol{k}} N = 0. \tag{3-13}$$

Now we want to calculate the response of N to the shear. So we set $N = \langle N(\boldsymbol{k}) \rangle + \tilde{N}$ and treat $\tilde{\boldsymbol{v}}_{\boldsymbol{E}}$ as a seed. In addition, we'd like to include interactions that don't conserved the action. In practice, we include them in an approximate form as a collisional damping on the right-hand side of Eq (3-11). Then we have

$$\frac{dN}{dt} = c(N) \backsimeq \gamma N - \Delta\omega N^2. \tag{3-14}$$

Just like usual game of quasi-linear theory, we plug $N = \langle N(\boldsymbol{k}) \rangle + \tilde{N}$ into Eq (3-14) and do the average, then we obtain

$$\begin{aligned} \frac{\partial \langle N \rangle}{\partial t} &= \frac{\partial}{\partial k_r}\langle k_\theta \tilde{v}_E' \tilde{N} \rangle + \langle c(N) \rangle \\ \frac{\partial \tilde{N}}{\partial t} + \boldsymbol{v}_{gr} \cdot \nabla \tilde{N} + |\gamma|\tilde{N} &= k_\theta \tilde{v}_E' \frac{\partial}{\partial k_r}\langle N \rangle \end{aligned}. \tag{3-15}$$

So

$$\tilde{N}_{q_r,\Omega} = \frac{i k_\theta q_r \tilde{v}_{Eq}}{-i(\Omega - q_r v_{gr} + i|\gamma|)} \frac{\partial \langle N \rangle}{\partial k_r}, \tag{3-16}$$

where q_r is the wave vector of perturbations in radial direction.

Substituting Eq (3-16) into Eq (3-15), we get

$$\frac{\partial \langle N \rangle}{\partial t} = \frac{\partial}{\partial k_r} D_k \frac{\partial}{\partial k_r}\langle N \rangle + \langle c(N) \rangle, \tag{3-17}$$

where

$$D_k = k_\theta^2 \sum_{q_r} \frac{q_r^2 \left|\tilde{v}_{E_q}\right|^2 |\gamma|^2}{(\Omega - q_r v_{gr})^2 + |\gamma|^2}. \tag{3-18}$$

The first term on the RHS of Eq (3-16) looks like a diffusion in k_r due to "random shearing". As $dk_r/dt = -\partial_x(k_\theta \tilde{v}_E')$ is in the form of Langevin equation, we expect $\langle k_r^2 \rangle$ will increase like a random walk. Here the irreversibility comes from ray chaos (i.e., entropy increases). Obviously, we can also approach via Envelope theory, i.e., Langmuir turbulence. The envelope equations can produce solitons in 1-dimension and collapse in 3-dimension. Both approaches can lead us to modulational instability by separation of time.

Based on the relation between $\langle N \rangle$ and $\langle \epsilon \rangle$, we have

$$\frac{\partial \langle \epsilon \rangle}{\partial t} = \int d^3k \omega_{\boldsymbol{k}} \frac{\partial}{\partial k_r} D_{\boldsymbol{k}} \frac{\partial}{\partial k_r} \langle N \rangle + \int d^3k \omega_{\boldsymbol{k}} \langle c(N) \rangle. \qquad (3-19)$$

By integration by parts, we can rewrite Eq (3-19) as

$$\frac{\partial \langle \epsilon \rangle}{\partial t} = -\int d^3k \frac{\partial \omega_{\boldsymbol{k}}}{\partial k_r} D_{\boldsymbol{k}} \frac{\partial}{\partial k_r} \langle N \rangle + \int d^3k \omega_{\boldsymbol{k}} \langle c(N) \rangle + S.T.. \qquad (3-20)$$

To determine the sign of $\partial_t \langle \epsilon \rangle$, we need to determine the signs of $\partial_{k_r} \omega$ and $\partial_{k_r} \langle N \rangle$. For EDW, the group velocity is

$$\frac{\partial \omega}{\partial k_r} = -\frac{2k_r k_\theta v_*}{(1+k_\perp^2 \rho_s^2)^2}. \qquad (3-21)$$

And the action is

$$N = \frac{\epsilon_{\boldsymbol{k}}}{\omega_{\boldsymbol{k}}} = \frac{(1+k_\perp^2 \rho_s^2)^2}{k_\theta v^*} |\phi_{\boldsymbol{k}}|^2 \qquad (3-22)$$

So

$$\frac{\partial \langle \epsilon \rangle}{\partial t} = \int d^3k \frac{2k_r}{(1+k_\perp^2 \rho_s^2)^2} \frac{\partial}{\partial k_r} |q_{\boldsymbol{k}}|^2 + \int d^3k \omega_{\boldsymbol{k}} \langle c(N) \rangle \qquad (3-23)$$

where $|q_{\boldsymbol{k}}|^2 = (1+k_\perp^2 \rho_s^2)^2 |\phi_{\boldsymbol{k}}|^2$.

Therefore, $\partial_t \langle \epsilon \rangle < 0$ as $\partial_{k_r} |q_{\boldsymbol{k}}|^2 < 0$ is always the case. So the energy flows from waves to zonal modes! To show the energy is conserved, we need another equation for zonal flow, which together with the wave kinetic equation forms the Predator-Prey model.